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AN ALTERNATIVE FORMULATION OF THE LIFTING LINE WING EQUATION AND ITS SOLUTION

by

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ABSTRACT:

In this paper the standard wing equation, as normally derived from lifting line theory, is further refined and a solution procedure more basic than the usual collocation technique is developed. The calculation method adopted avoids the necessity of performing an explicit matrix inversion; all equations can be solved sequentially, one at a time. On the other hand this technique involves the evaluation of numerous integrals over the span. The calculations are cumulative, and can be carried as far as necessary to achieve any required degree of accuracy. The analysis is interesting not only for purposes of practical calculation but also for the light it sheds on the essential mathematical structure of the basic aerodynamic phenomena involved. This same general method of calculation can also be readily adapted to the solution of other common types of engineering problems.

CONTENTS

1. INTRODUCTION

The standard wing equation, as usually derived on the basis of elementary lifting line theory, is normally solved by the collocation technique. While this approach gives acceptable results under normal conditions, circumstances can arise which make a somewhat deeper analysis desirable. The first purpose of this paper is to show that the standard wing equation itself can with advantage be further developed and refined. In particular, it is profitable to split this equation into two parts which respectively govern the basic and additional lift distributions, and to refine the resulting pair of equations by an integration process. The second purpose of the paper is to develop a calculation technique which is somewhat more fundamental and powerful than the usual collocation procedure. The method selected is based on the theory of orthogonal functions. It provides an explicit solution procedure which can be carried out as far as necessary to satisfy the initial equations to any required level of accuracy. The solution obtained is equivalent to that obtainable by matrix inversion, but the present technique achieves this result by an indirect process in which an explicit matrix inversion is not required at any point. The development of this method is of interest not only for purposes of practical calculation but also for the theoretical light it sheds on the mathematical structure of wing aerodynamics. The same general calculation technique can also be readily adapted to the calculation of other types of engineering problems

as well; the wing problem represents only one particular application of the method.

All symbols are defined in Section 2. For readers interested primarily in the resulting method of calculation, rather than in the detailed analysis itself, the calculation procedure is summarized in Section 3. The essentials of the standard wing equation are reviewed briefly in Section 4 and the suggested refinements are presented in Section 5. Solution by the usual collocation method is briefly discussed in Section 6. Orthogonal functions, which provide the basis for the proposed alternative method of solution, are analyzed in Section 7, and the resulting solution is developed in Section 8.

2. DEFINITIONS OF SYMBOLS

a	wing lift curve slope, $\frac{dC_L}{d\alpha}$
a_{nk}	coefficients which occur in Eq. (7.8)
a_∞	section lift curve slope
$\bar{a}_\infty$	mean section lift curve slope as defined in Eq. (3.4)
A_n	initially unknown constants which define the additional lift distribution, Eq. (3.34)
A^*_n	constants defined by Eq. (8.11)
AR	aspect ratio $2\ s/\bar{c}$
b_{nk}	elements in orthogonality matrix, Eq. (3.12)
B_n	initially unknown constants which define the basic lift distribution, Eq. (3.33)

c local wing chord at spanwise station η, ft.

$\bar{c}$ mean chord, $S/2s$, ft.

c_ℓ section lift coefficient at location η

C_n initially unknown constants which define the circulation function, Eq. (4.3)

C_L wing lift coefficient

C_{D_i} induced drag coefficient of wing

f forcing function defined in Eqs. (3.23) or (3.24)

F_n forcing constants defined in Eq. (3.25)

I_n planform integrals defined in Eq. (3.9)

J_{nk} wing integral defined in Eq (3.11)

K_{nk} integrals defined in Eq. (7.6)

s wing semi-span, ft.

S wing area, $ft.^2$

V_∞ remote velocity, ft/sec

w downwash velocity at wing station η, ft/sec

α angle of attack of wing, that is, angle between zero lift direction of mid-span section and V_∞, rad

α_{L_o} angle of attack of wing at which resultant wing lift vanishes, rad

α_t angle of twist, that is, angle between zero lift direction at station η and zero lift direction at mid-span, a prescribed function of η or θ, rad.

α_{t_1} twist angle at wing tip, rad

Γ circulation about wing at spanwise station η, ft^2/sec

$\left.\begin{array}{l}\delta_B \\ \delta_{BA} \\ \delta_A\end{array}\right\}$ Induced drag factors defined in Eqs. (3.30), (3.31) and (3.32)

δ_{nk} Kronecker delta, defined in Eq. (7.2)

η spanwise location expressed as fraction of semi-span

θ angular coordinate defined in Eq. (3.1)

$\left.\begin{array}{l}\lambda_B \\ \lambda_A\end{array}\right\}$ basic and additional lift functions defined in Eqs. (5.25) and (5.26)

λ_n orthogonal lift function defined in Eq. (8.23)

ν planform function defined in Eq. (3.6)

σ auxiliary parameter defined in Eq. (3.5)

τ twist function defined in Eq. (3.7)

$\bar{\tau}$ mean twist defined in Eq. (3.8)

τ_{L_o} relative angle of zero lift as defined in Eq. (3.27)

Φ_n orthogonal functions defined in Eqs. (7.2) and (7.4).

Ψ_n wing functions defined in Eq. (3.10)

Ψ_n' wing functions defined in Eq. (4.14)

$\left.\begin{array}{l}\omega_A \\ \omega_B\end{array}\right\}$ basic and additional downwash functions defined in Eqs. (5.27) and (5.28)

Subscripts

i, k, n indices which may take on the values 3, 5, 7 --- N

A pertains to additional lift

B pertains to basic lift

Matrix Symbols

$\left\{ \quad \right\}$ column vector containing elements 3, 5, 7 --- N

$\left\{ \quad \right\}_n$ column vector containing elements 3, 5, 7 --- n < N

$\left\{ \quad \right\}^T$ row vector containing elements 3, 5, 7, -- N

$\left[\quad \right]$ a square matrix with rows and columns corresponding to indices i = 3, 5, 7, --- N.

$\left[\quad \right]_n$ A square matrix with rows and columns corresponding to indices i = 3,5,7, --- n < N.

$\left[\ I \ \right]$ the identity matrix

3. SUMMARY OF THE CALCULATION METHOD

The proposed method of calculation is summarized in full in this section. Only such explanations are included here as are essential for making the method of calculation clear. All questions of derivation and interpretation are deferred to other sections of this paper. The formulas are listed in the approximate order in which they would be used in the calculations.

Preliminaries

$\eta = \cos\theta$ Change of variable (3.1)

$\bar{c} = \dfrac{S}{2s}$ Average chord (3.2)

$$AR = \frac{2s}{\bar{c}} \qquad \text{Aspect ratio} \tag{3.3}$$

$$\bar{a}_\infty = \int_0^1 a_\infty \left(\frac{c}{\bar{c}}\right) d\eta \qquad \text{Mean slope of section lift curve} \tag{3.4}$$

$$\sigma = \frac{\bar{a}_\infty}{\pi AR} \qquad \text{An auxiliary parameter} \tag{3.5}$$

$$\nu(\theta) = \left(\frac{a_\infty}{\bar{a}_\infty}\right)\left(\frac{c}{\bar{c}}\right) \qquad \text{Planform function} \tag{3.6}$$

$$\tau(\theta) = \left(\frac{\alpha_t}{\alpha_{t_1}}\right) \qquad \text{Twist function} \tag{3.7}$$

$$\bar{\tau} = \int_0^1 \tau \nu d\eta \qquad \text{Mean twist} \tag{3.8}$$

$$I_n = \int_0^{\pi/2} \nu \sin n\theta d\theta \qquad n = 3, 5, 7, \text{--} N \qquad \text{Planform integrals} \tag{3.9}$$

$$\Psi_n = \frac{4}{\pi} \sin n\theta + n\sigma\nu \left(\frac{\sin n\theta}{\sin\theta} - I_n\right) \qquad \text{Wing functions} \tag{3.10}$$

$n = 3, 5, 7, \text{---} N$

$$J_{nk} = \int_0^{\pi/2} \Psi_n \Psi_k d\theta \qquad n, k = 3, 5, 7 \text{---} N \qquad \text{Wing integrals} \tag{3.11}$$

Orthogonality Matrix

The matrix [b] is calculated as summarized below. It has the structure

$$[b] = \begin{bmatrix} b_{33} & 0 & 0 & -- & 0 \\ b_{53} & b_{55} & 0 & -- & 0 \\ b_{73} & b_{75} & b_{77} & -- & 0 \\ --- & --- & --- & -- & -- \\ B_{N3} & b_{N5} & b_{N7} & -- & b_{NN} \end{bmatrix} \qquad (3.12)$$

The first three rows are calculated as follows

<u>Row n = 3</u>

$$b_{33}^{-2} = J_{33} \qquad (3.13)$$

<u>Row n = 5</u>

$$a_{53} = -b_{33}\, J_{53} \qquad (3.14)$$

$$b_{55}^{-2} = J_{55} - a_{53}^{2} \qquad (3.15)$$

$$b_{53} = b_{55}\, a_{53}\, b_{33} \qquad (3.16)$$

<u>Row n = 7</u>

$$\begin{Bmatrix} a_{73} \\ a_{75} \end{Bmatrix} = - \begin{bmatrix} b_{33} & 0 \\ b_{53} & b_{55} \end{bmatrix} \begin{Bmatrix} J_{73} \\ J_{75} \end{Bmatrix} \qquad (3.17)$$

$$b_{77}^{-2} = J_{77} - \begin{Bmatrix} a_{73} \\ a_{75} \end{Bmatrix}^{T} \begin{Bmatrix} a_{73} \\ a_{75} \end{Bmatrix} \qquad (3.18)$$

$$\begin{Bmatrix} b_{73} \\ b_{75} \end{Bmatrix}^{T} = b_{77} \begin{Bmatrix} a_{73} \\ a_{75} \end{Bmatrix}^{T} \begin{bmatrix} b_{33} & 0 \\ b_{53} & b_{55} \end{bmatrix} \qquad (3.19)$$

Any desired additional number of rows can be calculated by repeated application of the following sequence, namely

Row n

$$\begin{Bmatrix} a_{n3} \\ a_{n5} \\ --- \\ a_{n,n-2} \end{Bmatrix} = - \begin{bmatrix} b_{33} & 0 & -- & 0 \\ b_{53} & b_{55} & -- & 0 \\ -- & -- & -- & 0 \\ b_{n-2,3} & b_{n-2,5} & -- & b_{n-2,n-2} \end{bmatrix} \begin{Bmatrix} J_{n3} \\ J_{n5} \\ -- \\ J_{n,\,n-2} \end{Bmatrix}, \tag{3.20}$$

$$b_{nn}^{-2} = J_{nn} - \begin{Bmatrix} a_{n3} \\ a_{n7} \\ --- \\ a_{n,n-2} \end{Bmatrix}^T \begin{Bmatrix} a_{n3} \\ a_{n7} \\ --- \\ a_{n,n-2} \end{Bmatrix}, \tag{3.21}$$

$$\begin{Bmatrix} b_{n3} \\ b_{n5} \\ --- \\ b_{n,n-2} \end{Bmatrix}^T = b_{nn} \begin{Bmatrix} a_{n3} \\ a_{n5} \\ -- \\ a_{n,n-2} \end{Bmatrix}^T \begin{bmatrix} b_{33} & 0 & -- & 0 \\ b_{53} & b_{55} & -- & 0 \\ -- & -- & -- & -- \\ b_{n-2,\,3} & b_{n-2,5} & -- & b_{n-2,n-2} \end{bmatrix} \tag{3.22}$$

Basic and Additional Lift

To calculate the additional lift distribution, set

$$f(\theta) = \upsilon(\theta) - \frac{4}{\pi} \sin\theta \qquad \text{Additional forcing function} \tag{3.23}$$

To calculate the basic lift distribution, set

$$f(\theta) = \nu(\theta)\left[\tau(\theta) - \bar{\tau}\right] \qquad \text{Basic forcing function} \tag{3.24}$$

and change the constants A_n below to B_n.

$$F_n = \int_0^{\pi/2} f(\theta)\,\psi_n(\theta)\,d\theta \qquad \text{Basic or additional forcing constants} \tag{3.25}$$

$$n = 3, 5, 7, \text{---}\ N$$

Solution Constants

$$\begin{Bmatrix} A_3 \\ A_5 \\ A_7 \\ \text{---} \\ A_N \end{Bmatrix} = [b]^T [b] \begin{Bmatrix} F_3 \\ F_5 \\ F_7 \\ \text{---} \\ F_N \end{Bmatrix} \tag{3.26}$$

Wing Characteristics

$$\frac{\alpha_{Lo}}{\alpha_{t_1}} = \tau_{L_o} = \sum_{n=3,5,7,\text{---}}^{N} nB_n I_n - \bar{\tau} \qquad \text{Zero lift angle} \tag{3.27}$$

$$\left(\frac{\bar{a}_\infty}{a}\right) = 1 + \sigma\left(1 + \sum_{n=3,5,7}^{N} nA_n I_n\right) \qquad \text{Wing lift curve slope} \tag{3.28}$$

$$C_{D_i} = \frac{1}{\pi AR}\left[(\bar{a}_\infty \alpha_{t_1})^2\,\delta_B + (\bar{a}_\infty \alpha_{t_1} C_L)\,\delta_{BA} + C_L^2\,(1 + \delta_A)\right] \tag{3.29}$$

Induced Drag

where

$$\delta_B = \sum_{n=3,5,7,--}^{N} nB_n^2 \tag{3.30}$$

$$\delta_{BA} = \sum_{n=3,5,7,--}^{N} nB_n A_n \tag{3.31}$$

$$\delta_A = \sum_{n=3,5,7,--}^{N} nA_n^2 \tag{3.32}$$

Induced drag factors.

Lift Distribution

$$\frac{c_{\ell B}}{\bar{a}_\infty \alpha_{t_1}}\left(\frac{c}{\bar{c}}\right) = \frac{4}{\pi} \sum_{n=3,5,7,--}^{N} B_n \sin n\theta \qquad \text{Basic} \tag{3.33}$$

$$\frac{c_\ell}{C_L}\left(\frac{c}{\bar{c}}\right) = \frac{4}{\pi} \sin\theta + \sum_{n=3,5,7,--}^{N} A_n \sin \eta\theta \qquad \text{Additional} \tag{3.34}$$

4. REVIEW OF THE STANDARD WING EQUATION

By utilizing the standard lifting line idealization of a wing, and employing the Biot Savart law, we can deduce [1, 2] * that the downwash w at any point η along the span is related to the distribution of circulation Γ along the span as follows:

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* Numbers indicate references listed at end of paper.

$$\frac{w}{V_\infty} = \frac{1}{\pi} \int_{-1}^{+1} \frac{-\frac{d}{d\eta'}\left(\frac{\Gamma}{4V_\infty s}\right) d\eta'}{(\eta' - \eta)} \tag{4.1}$$

It is customary to introduce the change of variable

$$\begin{aligned} \eta &= \cos\theta \\ \eta' &= \cos\theta' \end{aligned} \tag{4.2}$$

and to describe the distribution of the circulation Γ by a series of the form

$$\left(\frac{\Gamma}{4V_\infty s}\right) = \sum_{n=1,3,5,--}^{\infty} C_n \sin n\theta' \tag{4.3}$$

The present discussion is restricted to lift distributions which are symmetrical about the mid-span and hence only terms of odd index are retained in Eq. (4,3).

Upon introducing Eq. (4.3) into Eq. (4.1) and making use of the integral formula

$$\frac{1}{\pi} \int_0^{\pi} \frac{\cos n\theta' \, d\theta'}{\cos\theta' - \cos\theta} = \frac{\sin n\theta}{\sin\theta} \tag{4.4}$$

the following solution is obtained for the downwash

$$\left(\frac{w}{V_\infty}\right) = \sum_{n=1,3,5,---}^{\infty} C_n \frac{n \sin n\theta}{\sin\theta} \tag{4.5}$$

Once the above integration has been performed, there is no further need to distinguish between the variables η and η' or θ and θ' and the prime marks may be deleted from Eqs. (4.2) and (4.3).

The lift exerted by an element of the wing lying between η and $(\eta + d\eta)$ may be expressed firstly in terms of the section lift coefficient c_ℓ and secondly in terms of the circulation Γ, in accordance with the Kutta Joukowski law. Equating these two alternative expressions for the local lift gives

$$c_\ell \left(\frac{c}{\bar{c}} \right) = 4 \text{ Æ} \left(\frac{\Gamma}{4 V_\infty s} \right) \tag{4.6}$$

where

$$\bar{c} = \frac{S}{2s} = \text{average chord} \tag{4.7}$$

$$\text{Æ} = \frac{2s}{\bar{c}} = \text{aspect ratio} \tag{4.8}$$

The local lift coefficient c_ℓ can be expressed in terms of the local section lift curve slope a_∞ and the local net angle of attack, taking into account the effect of the local downwash angle $\left(\frac{w}{V_\infty} \right)$ in reducing the effective angle of attack. Hence we obtain

$$c_\ell = a_\infty \left(\alpha + \alpha_t - \frac{w}{V_\infty} \right) \tag{4.9}$$

where α = wing angle of attack = angle of attack of the zero lift direction of the airfoil section at midspan with respect to the remote velocity V_∞.

and $\alpha_t(\theta)$ = angle of twist = angle of zero lift direction at η with respect to zero lift direction at midspan, a prescribed function of θ.

Upon combining Eqs. (4.3), (4.5), (4.6) and (4.9) and rearranging we can finally obtain the fundamental wing equation in the following form, namely,

$$\sum_{n=1,3,5,----} C_n \psi'_n(\theta) = \sigma \nu (\alpha + \alpha_t) \tag{4.10}$$

where

$$\bar{a}_\infty = \int_0^1 a_\infty \left(\frac{c}{\bar{c}}\right) d\eta \quad = \quad \text{mean section lift curve slope} \tag{4.11}$$

$$\sigma = \frac{\bar{a}_\infty}{\pi AR} \quad = \quad \text{an auxiliary parameter} \tag{4.12}$$

$$\nu(\theta) = \left(\frac{a_\infty}{\bar{a}_\infty}\right)\left(\frac{c}{\bar{c}}\right) = \quad \text{a prescribed planform function} \tag{4.13}$$

$$\psi'_n(\theta) = \frac{4}{\pi} \sin n\theta + \sigma\nu \frac{n \sin n\theta}{\sin\theta} \tag{4.14}$$

Apart from minor changes in notation, Eq. (4.10) is the fundamental wing equation as found in most elementary texts on aerodynamics [1,2].

5. ALTERNATIVE EQUATIONS FOR BASIC AND ADDITIONAL LIFT

By substituting Eq. (4.3) into (4.6) multiplying through by $d\eta = -\sin\theta\, d\theta$, integrating over the semi-span and taking advantage of the orthogonality relation

$$\int_0^{\pi/2} \sin n\theta \sin k\theta\, d\theta = \delta_{nk} \begin{array}{l} = +1 \text{ if } n = k \\ = 0 \text{ if } n \neq k \end{array} \tag{5.1}$$

we obtain a solution for the overall lift of the wing in the form

$$C_L = \int_0^1 c_\ell \left(\frac{c}{\bar{c}}\right) d\eta = \pi \text{AR}\, C_1 \tag{5.2}$$

This verifies the well known fact that the overall lift of the wing is fixed solely by the first term of the series in Eq. (4.3). The other terms affect the form of the lift distribution but not the resultant net lift.

It is useful to express the twist distribution of the wing in dimensionless form as follows

$$\tau(\theta) = \left(\frac{\alpha_t}{\alpha_{t_1}}\right) = \text{a prescribed twist function} \tag{5.3}$$

where α_{t_1} = angle of twist at wing tip (5.4)

The basic wing relation, Eq. (4.14), can be further developed by introducing the substitution

$$C_n = \frac{1}{\pi AR}\left[a_\infty \alpha_{t_1} B_n + C_L A_n\right] \tag{5.5}$$

$n = 1, 3, 5, ---$

The constants B_n now define the basic lift distribution of the wing. This is the distribution associated with the wing twist. Specifically it is the distribution which occurs when the wing is at the angle of attack $\alpha = \alpha_{L_o}$ which gives $C_L = 0$. The constants An define the additional lift distribution that occurs as a result of changes in angle of attack. Hence it is proportional to the angle $(\alpha - \alpha_{Lo})$ or to C_L itself.

Consider Eq. (5.5) for the specific case n = 1. Substitute for C_1 from Eq. (5.2). The resulting expression must be satisfied for all values of C_L. In particular by setting $C_L = 0$ we find that

$$B_1 = 0$$

$$A_1 = 1 \tag{5.6}$$

Substituting the expressions (5.5) into the basic wing relation Eq. (4.10), and making use of the special result (5.6), we obtain

$$\sigma \alpha_{t_1} \sum_{n=3,5,7,---} B_n \Psi'_n(\theta) + \frac{C_L}{\pi AR}\left[\sin\theta + \sum_{n=3,5,7,---} A_n \Psi'_n(\theta)\right] = \sigma \upsilon (\alpha + \alpha_t) \tag{5.7}$$

This must be satisfied for all operating conditions. In particular for $\alpha = \alpha_{L_0}$ and $C_L = 0$, it reduces to

$$\sigma \alpha_{t_1} \sum_{n=3,5,7,---} B_n \Psi'_n(\theta) = \sigma \upsilon (\alpha_{L_0} + \alpha_t) \tag{5.8}$$

Then subtracting Eq. (5.8) from (5.7) we obtain

$$\frac{C_L}{\pi AR}\left[\sin\theta + \sum_{n=3,5,7,---} A_n \Psi'_n(\theta) \right] = \sigma \upsilon (\alpha - \alpha_{L_0}) \tag{5.9}$$

Eq. (5.8) can be further reduced by dividing through by $\sigma \alpha_{t_1}$ and setting

$$\left(\frac{\alpha_{L_0}}{\alpha_{t_1}} \right) = \tau_{L0} \tag{5.10}$$

Eq. (5.9) can be further reduced by substituting

$$C_L = a(\alpha - \alpha_{L0}) \tag{5.11}$$

$$\text{where} \quad a = \left(\frac{dC_L}{d\alpha} \right) = \text{wing lift curve slope} \tag{5.12}$$

In this way we obtain for the basic and additional lift the two fundamental equations.

$$\sum_{n=3,5,7,---} B_n \Psi'_n(\theta) = \upsilon(\tau_{L_0} + \tau) \tag{5.13}$$

$$\sin\theta + \sum_{n=3,5,7,---} A_n \Psi'_n(\theta) = \left(\frac{\bar{a}_\infty}{a} \right) \upsilon \tag{5.14}$$

The complete analysis of a twisted wing can be obtained either by solving the standard wing relation, Eq. (4.10) for two different angles of attack, or alternatively, by solving the above pair of relations, Eqs. (5.13) and (5.14), for basic and additional lift. The total number of unknowns to be found is exactly the same by both methods; only the form is different. It can be said, however, that Eqs.(5.13) and (5.14) are somewhat more fundamental in form in the sense that, for a wing of prescribed geometry, the unknowns B_n, τ_0, A_n and $\left(\frac{\bar{a}_\infty}{a}\right)$ are true invariants, whereas each one of the "constants" C_n in Eq. (4.10) is actually a function of angle of attack.

Moreover, Eqs. (5.13) and (5.14) lend themselves to further development in a useful and interesting way. In this connection, the first step is to multiply these equations through by $d\eta = -\sin\theta d\theta$ and integrate over the semispan.

This operation involves the following integrals, Firstly,

$$\int_0^1 \nu \, d\eta = \int_0^1 \left(\frac{a_\infty}{\bar{a}_\infty}\right)\left(\frac{c}{\bar{c}}\right) d\eta = 1 \qquad (5.15)$$

a result which follows directly from the definition of $\bar{a}_\infty$ as given in Eq. (4.11).

Secondly,

$$\frac{4}{\pi}\int_0^1 \sin n\theta \, d\eta = \frac{4}{\pi}\int_0^{\pi/2} \sin n\theta \sin\theta d\theta = 1 \text{ for } n = 1$$
$$= 0 \text{ for } n \neq 1 \qquad (5.16)$$

Thirdly, we define the average value $\bar{\tau}$ of the twist function by the integral

$$\bar{\tau} = \int_0^1 \tau \upsilon d\eta = \int_0^{\pi/2} \tau \upsilon \sin \theta d\theta \tag{5.17}$$

Finally, we define the series of integrals

$$I_n = \int_0^1 \upsilon \frac{\sin n\theta}{\sin \theta} d\eta = \int_0^{\pi/2} \upsilon \sin n\theta d\theta \tag{5.18}$$

$$n = 1,3,5, \text{-----}$$

We also note from this last definition that specifically for n = 1

$$I_1 = \int_0^1 \upsilon d\eta = 1 \tag{5.19}$$

Upon integrating Eqs. (5.13) and (5.14) over the semi-span, utilizing the foregoing integrals, then simplifying, we obtain the two significant results below, namely,

$$\tau_{L_0} = \sum nB_n I_n - \bar{\tau} \tag{5.20}$$

$$n = 3,5,7, \text{----}$$

$$\frac{\bar{a}_\infty}{a} = 1 + \sigma\left[1 + \sum' nA_n I_n\right] \tag{5.21}$$

$$n = 3,5,7,----$$

These results are useful for two reasons. First of all, they can be used to eliminate the unknowns τ_{L_0} and $\left(\frac{\bar{a}_\infty}{a}\right)$ from Eqs. (5.13) and (5.14) thus reducing the number of simultaneous unknowns which must be found from these equations. Secondly, they provide a subsequent solution for these two unknowns, and one that is especially accurate because it is based on an integration over the entire wing.

After Eqs. (5.20) and (5.21) have been used to eliminate τ_{L_o} and $\left(\frac{\bar{a}_\infty}{a}\right)$ from Eqs. (5.13) and (5.14), and after the results have been simplified, they become finally

$$\sum_{n = 3,5,7,----} B_n \psi_n(\theta) = \upsilon(\tau - \bar{\tau}) \tag{5.22}$$

$$\sum_{n = 3,5,7,----} A_n \psi_n(\theta) = \left(\upsilon - \frac{4}{\pi} \sin \theta\right) \tag{5.23}$$

where

$$\psi_n(\theta) = \frac{4}{\pi} \sin n\theta + n\sigma\upsilon\left(\frac{\sin n\theta}{\sin \theta} - I_n\right) \tag{5.24}$$

$$n = 3,5,7,----$$

Eqs. (5.22) and (5.23) represent an alternative form of the fundamental wing equations. They can advantageously replace the standard

wing equation (4.10) as usually employed in the lifting line analysis. The series on the left side of Eqs. (4.10) and (5.22) and (5.23) are, of course, truncated at some value $n = N$, depending on the accuracy required. Often only the four terms in C_1, C_3, C_5 and C_7 are retained in Eq. (4.10). Note that in this case, Eqs. (5.22) and (5.23) will contain only three unknowns apiece, namely, B_3, B_5, B_7 and A_3, A_5. A_7. Moreover the latter solutions can be expected to be somewhat more accurate since they include the averaging effects of integration over the semi-span.

Note that Eqs. (5.22) and (5.23) are of identical format on the left. Only the forcing functions on the right are different. Eq. (5.22) shows clearly that the constants B_n depend only on the dimensionless planform and twist functions and, in particular, that an untwisted wing gives $B_n = 0$. Likewise, Eq. (5.23) shows that the constants A_n depend only on the deviation of the planform function υ from an ellipse of equal area and, in particular, that a wing having an elliptical planform function gives $A_n = 0$.

It should also be noted that while Eqs. (5.22) and (5.23) are more effective than Eq. (4.10) in the sense noted above, they entail the additional labor of evaluating the integrals $\bar{\tau}$ and I_n.

After Eqs. (5.22) and (5.23) have been solved for the constants B_n and A_n, the angle of zero lift $\left(\frac{\alpha_{L_0}}{\alpha_{t_1}}\right) = \tau_{L_0}$ and the slope of the wing lift curve $\left(\frac{a}{a_\infty}\right)$ can then be found accurately from Eqs. (5.20) and (5.21), respectively.

The resulting basic and additional lift and downwash distributions can be found by combining Eqs. (4.3), (4.5), (4.6) and (5.5). We thereby obtain the following solutions.

$$\left(\frac{c_{\ell B}}{a_\infty \alpha_{t_1}}\right)\left(\frac{c}{\bar{c}}\right) = \frac{4}{\pi} \sum_{n=3,5,7,----} B_n \sin n\theta = \lambda_B(\theta) \tag{5.25}$$

$$\frac{c_{\ell A}}{C_L}\left(\frac{c}{\bar{c}}\right) = \frac{4}{\pi}\left[\sin\theta + \sum_{n=3,5,7,--} A_n \sin n\theta\right] = \lambda_A(\theta) \tag{5.26}$$

$$\left(\frac{\pi AR}{a_\infty \alpha_{t_1}}\right)\left(\frac{w_B}{V_\infty}\right) = \sum_{n=3,5,7,----} B_n \frac{n \sin n\theta}{\sin\theta} = \omega_B(\theta) \tag{5.27}$$

$$\frac{\pi AR}{C_L}\left(\frac{w_A}{V_\infty}\right) = 1 + \sum_{n=3,5,7,----} A_n \frac{n \sin n\theta}{\sin\theta} = \omega_A(\theta) \tag{5.28}$$

Once the constants A_n and B_n are known, they fix the four functions above which define the fundamental aerodynamics of the particular wing under consideration.

The one remaining question of basic significance which falls within the purview of lifting line theory concerns the distribution and overall value of the induced drag. The overall coefficient of induced drag is defined by the integral

$$C_{Di} = \int_0^1 \left(c_\ell \frac{c}{\bar{c}}\right)\left(\frac{w}{V_\infty}\right) d\eta = \int_0^1 \left[c_{\ell_B}\frac{c}{\bar{c}} + c_{\ell_A}\frac{c}{\bar{c}}\right]\left[\frac{w_B}{V_\infty} + \frac{w_A}{V_\infty}\right] d\eta \tag{5.29}$$

Upon substituting for the quantities in Eq. (5.29) in terms of the four functions defined in Eqs. (5.25) through (5.28), then rearranging, we obtain

$$C_{D_i} = \frac{\left(\bar{a}_\infty \alpha_{t_1}\right)^2}{\pi AR} \int_0^{\pi/2} \lambda_B(\theta)\, \omega_B(\theta) \sin\theta d\theta$$

$$+ \frac{\bar{a}_\infty \alpha_{t_1} C_L}{\pi AR} \int_0^{\pi/2} \left[\lambda_B(\theta)\, \omega_A(\theta) + \lambda_A(\theta)\, \omega_B(\theta)\right] \sin\theta d\theta$$

$$+ \frac{C_L^2}{\pi AR} \int_0^{\pi/2} \lambda_A(\theta)\, \omega_A(\theta) \sin\theta d\theta \qquad (5.30)$$

The integrals on the right side of Eq. (5.30) show that there are three distinct components of induced drag and the respective integrands define the corresponding distributions. However, instead of a direct evaluation of these three separate integrals, it is simpler to proceed as follows. First we evaluate the overall integral in Eq. (5.29) using Eqs. (4.3), (4.5) and (4.6). The result is simply

$$C_{D_i} = \pi AR \sum_{n=1,3,5,----} nC_n^2 \qquad (5.31)$$

Next we substitute Eq. (5.5) into (5.31), expand and regroup terms. In this way we obtain finally

$$C_{D_i} = \frac{1}{\pi AR}\left[\left(\bar{a}_\infty \alpha_{t_1}\right)^2 \delta_B + \left(\bar{a}_\infty \alpha_{t_1} C_L\right)\delta_{BA} + C_L^2 (1 + \delta_A)\right] \tag{5.32}$$

where

$$\delta_B = \sum_{n=3,5,7,---} nB_b^2 \tag{5.33}$$

$$\delta_{BA} = \sum_{n=3,5,7,---} nB_n A_n \tag{5.34}$$

$$\delta_A = \sum_{n=3,5,7,---} nA_n^2 \tag{5.35}$$

We also note in passing that for the special case of an aerodynamically untwisted wing of elliptical planform, Eq. (5.26), (5.28), (5.21), and (5.32) reduce, respectively, to the following familiar and elementary forms, namely,

$$c_\ell\left(\frac{c}{\bar{c}}\right) = C_L \frac{4}{\pi} \sin\theta = \text{elliptical lift distribution} \tag{5.36}$$

$$\frac{w}{V_\infty} = \frac{C_L}{\pi AR} = \text{uniform downwash along span} \tag{5.37}$$

$$a = \frac{\bar{a}_\infty}{1 + \frac{\bar{a}_\infty}{\pi AR}} = \text{slope of wing lift curve (a maximum)} \tag{5.38}$$

$$C_{D_i} = \frac{C_L^2}{\pi AR} = \text{induced drag (a minimum)} \tag{5.39}$$

These last four results illustrate the fact that the shift from the standard form (4.10) of the wing equation to the revised forms (5.22) and (5.23) does not affect the essential character of the solution obtained.

6. THE COLLOCATION METHOD

The usual method of obtaining an approximate solution of the standard wing relation, Eq. (4.10) is by collocation, that is, by satisfying this equation exactly only at a limited number of discrete points along the semi-span. The number of points must of course equal the number of unknown constants C_1, C_3, C_5, --- C_N to be retained in the approximate solution. The location of the collocation points is somewhat arbitrary, but it is customary to space them at equal intervals of θ along the semi-span, to include the midspan point at $\theta = \frac{\pi}{2}$, and to omit the wingtip point $\theta = 0$. The solution of the resulting set of simultaneous equations entails a matrix inversion, but is otherwise quite straightforward. If a computer be utilized, of course, even the matrix inversion poses no appreciable computational burden.

In practice the above calculations even if performed only on a desk calculator, usually work out in a reasonably satisfactory manner. This is partly owing to the fact that any well designed wing of moderate to high aspect ratio has a lift distribution that does not deviate strongly from the optimum elliptical distribution. Consequently the successive constants

C_1, C_3, C_5, --- typically diminish rapidly in magnitude, and only a few terms, often just four, suffice to produce results of an accuracy adequate for many engineering purposes. Moreover, a quest for extreme numerical accuracy would not be realistic because the basic lifting line idealization on which the wing equation is based is itself an approximation.

Nevertheless, there are situations in which the mathematical circumstances are not so favorable, for example, in cases of strong discontinuities in the lift distribution owing, say, to the action of wing flaps over part of the span. Moreover, apart from practical questions of computation, collocation has a number of features which are not optimal from the theoretical viewpoint. For example, the choice of the actual collocation points introduces an element of arbitrariness into the solution. It would be in some respects preferable to have a method whose accuracy depends in a known way only on the number of terms retained in the solution, and not on such contingencies as the location of collocation points. Furthermore, the collocation method can under certain adverse circumstances lead to a set of equations which are numerically ill conditioned. Moreover, the purely theoretical conception of how the collocation solution approaches its limit as the number of terms retained in the solution hypothetically increases without bound tends to be somewhat obscure; it entails the conceptual difficulty of visualizing the inversion of a matrix in the limit as the number of rows and columns increases without bound.

7. ORTHOGONAL FUNCTIONS

There is an alternative to the collocation method that deals with the above difficulties, both conceptual and computational, in a very clear and decisive way. It can be developed through the introduction of a certain family of orthogonal functions $\Phi_n(\theta)$, $n = 3,5,7,$ --- that are related in a simple way to the basic functions $\psi_n(\theta)$ that occur in the wing equation. The mathematical structure of these auxiliary functions is developed in this section. It turns out that the functions Φ_n, while essential to the theoretical development, can be eliminated from the final calculations; only certain constants that occur in the definition of these functions carry over into the resulting calculation procedure.

The auxiliary functions $\Phi_n(\theta)$ are related to the original functions $\psi_n(\theta)$ by equations of the following form:

$$\Phi_3(\theta) = b_{33}\,\psi_3$$

$$\Phi_5(\theta) = b_{53}\,\psi_3 + b_{55}\,\psi_5 \tag{7.1a}$$

$$\Phi_7(\theta) = b_{73}\,\psi_3 + b_{75}\,\psi_5 + b_{77}\,\psi_7$$

- -

$$\Phi_n(\theta) = b_{n3}\,\psi_3 + b_{n5}\,\psi_5 + \text{-------} + b_{nn}\,\psi_n \tag{7.1b}$$

The constant coefficients b_{nk} in these equations are chosen so that the functions $\Phi_n(\theta)$ form an ortho-normal set, that is, so that

$$\int_0^{\pi/2} \Phi_n \Phi_k \, d\theta = \delta_{nk} = 0 \quad \text{for } k = 3,5,7, \text{---} (n-2) \tag{7.2a}$$

$$= +1 \quad \text{for } k = n \tag{7.2b}$$

It is convenient to develop much of the theory in matrix format. Accordingly, we introduce the following abbreviated matrix notation.

$$\left\{\psi\right\}_n = \begin{Bmatrix} \psi_3(\theta) \\ \psi_5(\theta) \\ \psi_7(\theta) \\ \text{----} \\ \psi_n(\theta) \end{Bmatrix} \qquad \left\{\Phi\right\}_n = \begin{Bmatrix} \Phi_3(\theta) \\ \Phi_5(\theta) \\ \Phi_7(\theta) \\ \text{---} \\ \Phi_n(\theta) \end{Bmatrix}$$

$$[b]_n = \begin{bmatrix} b_{33} & 0 & 0 & \text{--} & 0 \\ b_{53} & b_{55} & 0 & \text{--} & 0 \\ b_{73} & b_{75} & b_{77} & \text{--} & 0 \\ \text{--} & \text{--} & \text{---} & \text{---} & \text{--} \\ b_{n3} & b_{n5} & b_{n7} & \text{--} & b_{nn} \end{bmatrix} \tag{7.3}$$

Consequently, Eqs. (7.1) may be written simply as

$$\left\{\Phi\right\}_n = [b]_n \left\{\psi\right\}_n \tag{7.4}$$

where we call [b] the orthogonality matrix.

In connection with the ensuing analysis, it will also be convenient to define the following auxiliary integrals, namely,

$$J_{nk} = J_{kn} = \int_0^{\pi/2} \psi_n(\theta)\, \psi_k(\theta)\, d\theta \qquad (7.5)$$

and

$$K_{nk} = \int_0^{\pi/2} \psi_n(\theta)\, \Phi_k(\theta)\, d\theta \qquad k = 3,5,7,---(n-2) \qquad (7.6)$$

The integrals K_{nk} will ultimately be eliminated from the analysis. The integrals J_{nk}, however, can be assembled to form the following matrix.

$$[J]_n = [J]_n^T = \begin{bmatrix} J_{33} & J_{35} & J_{37} & -- & J_{3n} \\ J_{53} & J_{55} & J_{57} & -- & J_{5n} \\ J_{73} & J_{75} & J_{77} & -- & J_{7n} \\ --- & --- & --- & -- & --- \\ J_{n3} & N_{n5} & N_{n7} & -- & J_{nn} \end{bmatrix} \qquad (7.7)$$

It can be seen that the matrix $[b]_{n-2}$ fixes the functions Φ_3, Φ_5, Φ_7, --- Φ_{n-2}. Now suppose that this matrix is known and that we wish to add to it the initially unknown n^{th} row of coefficients which will define the next function Φ_n. For this purpose, it is useful to write Φ_n temporarily in the alternative form

$$\Phi_n = b_{nn} \left[\sum_{k=3,5,7,----}^{n-2} a_{nk}\, \Phi_k + \psi_n \right] \qquad (7.8)$$

In matrix notation, this may be written in either of the two equivalent forms below, namely,

$$\Phi_n = b_{nn}\left(\left\{a_n\right\}^T_{n-2}\left\{\Phi\right\}_{n-2} + \psi_n\right) \tag{7.9}$$

$$\Phi_n = b_{nn}\left(\left\{\Phi\right\}^T_{n-2}\left\{a_n\right\}_{n-2} + \psi_n\right) \tag{7.10}$$

where

$$\left\{a_n\right\}_{n-2} = \begin{Bmatrix} a_{n3} \\ a_{n5} \\ a_{n7} \\ --- \\ a_{n,n-2} \end{Bmatrix} \tag{7.11}$$

Also, from (7.4)

$$\left\{\Phi\right\}^T_{n-2} = \left\{\psi\right\}^T_{n-2} [b]^T_{n-2} \tag{7.12}$$

Now premultiplying Eq. (7.10) through by $\left\{\Phi\right\}_{n-2}$ gives

$$\left\{\Phi\right\}_{n-2}\Phi_n = b_{nn}\left(\left\{\Phi\right\}_{n-2}\left\{\Phi\right\}^T_{n-2}\left\{a_n\right\}_{n-2} + \left\{\Phi\right\}_{n-2}\psi_n\right) \tag{7.13}$$

We now integrate this over the semi-span, that is, from $\theta = 0$ to $\theta = \frac{\pi}{2}$, then apply the ortho-normal condition (7.2) and the definitiions (7.5)

and (7.6). The following simplifications occur, namely,

$$\int_0^{\pi/2} \left\{ \Phi \right\}_{n-2} \Phi_n \, d\theta = 0 \tag{7.14}$$

$$\int_0^{\pi/2} \left\{ \Phi \right\}_{n-2} \left\{ \Phi \right\}_{n-2}^T d\theta = [I]_{n-2} = \begin{bmatrix} 1 & 0 & 0 & -- & 0 \\ 0 & 1 & 0 & -- & 0 \\ 0 & 0 & 1 & -- & 0 \\ -- & -- & -- & -- & -- \\ 0 & 0 & 0 & -- & 1 \end{bmatrix} \tag{7.15}$$

where $[I]_{n-2}$ is the identity matrix, with rows and columns corresponding to indices 3, 5, 7, --- (n-2).

Also

$$\int_0^{\pi/2} \left\{ \Phi \right\}_{n-2} \psi_n \, d\theta = \left\{ K_n \right\}_{n-2} = \begin{Bmatrix} K_{n3} \\ K_{n5} \\ K_{n7} \\ -- \\ K_{n,n-2} \end{Bmatrix} \tag{7.16}$$

With the aid of relations (7.14) (7.15) and (7.16), the integrated form of Eq. (7.13) can be rearranged to yield the following significant result.

$$[I]_{n-2} \left\{ a_n \right\}_{n-2} = \left\{ a_n \right\}_{n-2} = - \left\{ K_n \right\}_{n-2} \tag{7.17}$$

This defines the solution for the constants a_{nk} such as to satisfy the orthogonality conditions (7.2a). However, a further reduction of the right side of Eq. (7.17) is possible and useful. Note that, from the definition (7.6) and from (7.4),

$$\left\{K_n\right\}_{n-2} = \int_0^{\pi/2} \psi_n \left\{\Phi\right\}_{n-2} d\theta = \int_0^{\pi/2} \psi_n \; [b]_{n-2} \left\{\psi\right\}_{n-2} d\theta$$

$$= [b]_{n-2} \left\{J_n\right\}_{n-2} = [b]_{n-2} \begin{Bmatrix} J_{n3} \\ J_{n5} \\ J_{n7} \\ --- \\ J_{n,n-2} \end{Bmatrix} \tag{7.18}$$

Combining (7.17) and (7.18) gives the orthogonality relation in the desired final form, namely,

$$\left\{a_n\right\}_{n-2} = - [b]_{n-2} \left\{J_n\right\}_{n-2} \tag{7.19}$$

To apply the normalizing conditions (7.2b), we first post-multiply (7.9) by (7.10), then multiply through by b_{nn}^{-2}. This gives

$$b_{nn}^{-2} \Phi_n^2 = \left\{a_n\right\}_{n-2}^T \left\{\Phi\right\}_{n-2} \left\{\Phi\right\}_{n-2}^T \left\{a_n\right\}_{n-2} + \left\{a_n\right\}_{n-2}^T \left\{\Phi\right\}_{n-2} \psi_n$$

$$+ \psi_n \left\{\Phi\right\}_{n-2}^T \left\{a_n\right\}_{n-2} + \psi_n^2 \tag{7.20}$$

We next integrate this over the semi-span noting that, from (7.2b),

$$\int_0^{\pi/2} \Phi_n^2 \, d\theta = 1 \tag{7.21}$$

and from (7.5)

$$\int_0^{\pi/2} \psi_n^2 \, d\theta = J_{nn} \tag{7.22}$$

Also, the first term on the right side of (7.20) may be simplified after integration in the manner of (7.15). The next two terms, which incidentally are of equal magnitude, can be simplified according to (7.16). In this way we ultimately obtain from the integrated form of (7.20) the result

$$b_{nn}^{-2} = \left\{ a_n \right\}_{n-2}^{T} \left[\; I \; \right]_{n-2} \left\{ a_n \right\}_{n-2} + \left\{ a_n \right\}_{n-2}^{T} \left\{ K_n \right\}_{n-2}$$

$$+ \left\{ K_n \right\}_{n-2}^{T} \left\{ a_n \right\}_{n-2} + J_{nn} \tag{7.23}$$

This reduces readily to

$$b_{nn}^{-2} = \left\{ a_n \right\}_{n-2}^{T} \left\{ a_n \right\}_{n-2} + 2 \left\{ a_n \right\}_{n-2}^{T} \left\{ K_n \right\}_{n-2} + J_{nn} \tag{7.24}$$

We can eliminate $\left\{ K_n \right\}_{n-2}$ from this by using the orthogonality relation in the form given in (7.17). The final result therefore becomes simply

$$b_{nn}^{-2} = J_{nn} - \left\{ a_n \right\}_{n-2}^{T} \left\{ a_n \right\}_{n-2} \tag{7.25}$$

At this point the three relations (7.10), (7.19) and (7.25) together define the function Φ_n in such a way that it satisfies the normalizing condition (7.2b) and is orthogonal to all the previous functions in the set, namely, to Φ_3, Φ_5, Φ_7, --- Φ_{n-2}. However, it still remains to determine the coefficients b_{n3}, b_{n5}, b_{n7} --- $b_{n,\,n-2}$ which are needed to describe Φ_n in the standard format of Eq. (7.4). In this connection we note first that

$$\Phi_n = \begin{Bmatrix} b_{n3} \\ b_{n5} \\ b_{n7} \\ -- \\ b_{nn} \end{Bmatrix}^{T} \begin{Bmatrix} \psi_3 \\ \psi_5 \\ \psi_7 \\ --- \\ \psi_n \end{Bmatrix} = \left\{ b_n \right\}_{n}^{T} \left\{ \psi \right\}_{n}$$

$$= \left\{ b_n \right\}_{n-2}^{T} \left\{ \psi \right\}_{n-2} + b_{nn} \psi_n \tag{7.26}$$

Now a comparison of (7.26) with (7.9) shows that

$$\left\{ b_n \right\}_{n-2}^{T} \left\{ \psi \right\}_{n-2} = b_{nn} \left\{ a_n \right\}_{n-2}^{T} \left\{ \Phi \right\}_{n-2} \tag{7.27}$$

Moreover, by analogy with Eq. (7.4) we may write

$$\left\{ \Phi \right\}_{n-2} = [b]_{n-2} \left\{ \psi \right\}_{n-2} \tag{7.28}$$

Substitution of (7.28) into (7.27) discloses finally that

$$\left\{ b_n \right\}_{n-2}^T = b_{nn} \left\{ a_n \right\}_{n-2}^T [b]_{n-2} \tag{7.29}$$

This is the required result which, along with (7.19) and (7.25), enables the n^{th} row of matrix $[b]_n$ to be calculated when all the previous rows are known.

The principal results of this section can now be summarized. It is supposed that all quantities over the range of the index i = 3,5,7, --- (n-2) are known from previous calculation. The following calculations then extend these results to the additional index i = n.

Firstly, we must evaluate the additional integrals

$$J_{nk} = \int_0^{\pi/2} \psi_n(\theta)\ \psi_k(\theta)\, d\theta \qquad k = 3,5,7,\text{---}n \tag{7.29}$$

For any given wing planform, the functions ψ_n as defined by Eqs. (5.24) are known. Hence the constants J_{nk} are computable by direct numerical integration.

Next we calculate, in the order given, the quantities

$$\{a_n\}_{n-2} = -[b]_{n-2}\{J_n\}_{n-2} \tag{7.30}$$

$$b_{nn}^{-2} = J_{nn} - \{a_n\}_{n-2}^T\{a_n\}_{n-2} \tag{7.31}$$

$$\{b_n\}_{n-2}^T = b_{nn}\{a_n\}_{n-2}^T[b]_{n-2} \tag{7.32}$$

Eqs. (7.31) and (7.32) fix row n of the orthogonality matrix $[b]_n$. Consequently, the function Φ_n is now defined in the standard format, namely,

$$\Phi_n = \{b_n\}_n^T\{\Psi\}_n \tag{7.33}$$

Moreover, the function Φ_n so defined satisfies the ortho-normal relation (7.2) with respect to all previous functions in the set.

The above calculation sequence, Eq. (7.29) through (7.32), may then be repeated for the next successive row of the matrix [b]. In this way, the fundamental matrix [b] can be expanded to any desired number of rows and columns.

We shall next show how, once the orthogonality matrix has been established to any desired size, all remaining unknowns of the problem can be calculated to the corresponding degree of accuracy in a routine way.

8. SOLUTION BY MEANS OF THE ORTHOGONALITY MATRIX

Both of the revised wing relations, Eqs. (5.22) and (5.23), are solved by the same general method and hence for the purposes of this section, it is preferable to adopt a generalized notation which embraces both equations. Also, it is helpful to retain the matrix notation. Hence we rewrite Eqs. (5.22) and/or (5.23) simply as

$$\left\{ \psi \right\}^T \left\{ A \right\} = f(\theta) \tag{8.1}$$

where

$$\left\{ \psi \right\}^T = \left\{ \begin{matrix} \psi_3(\theta) \\ \psi_5(\theta) \\ \psi_7(\theta) \\ --- \\ \psi_N(\theta) \end{matrix} \right\}^T \quad \text{and} \quad \left\{ A \right\} = \left\{ \begin{matrix} A_3 \\ A_5 \\ A_7 \\ --- \\ A_N \end{matrix} \right\} \tag{8.2}$$

For calculating the additional lift distribution we set

$$f(\theta) = f_A(\theta) = \upsilon - \frac{4}{\pi} \sin\theta \tag{8.3}$$

For calculating the basic lift distribution we set

$$f(\theta) = f_B(\theta) = \upsilon\,(\tau - \overline{\tau}) \tag{8.4}$$

In this latter case we also change $\left\{ A \right\}$ to $\left\{ B \right\}$ for consistency with the notation established earlier.

It is also convenient to define the set of integral constants

$$F_n = \int_0^{\pi/2} f(\theta)\, \psi_n(\theta)\, d\theta \qquad n = 3, 5, 7, --- N \tag{8.5}$$

and, of course, the corresponding vector

$$\left\{ F \right\} = \left\{ \begin{array}{c} F_3 \\ F_5 \\ F_7 \\ -- \\ F_n \end{array} \right\} \tag{8.6}$$

Also since

$$\left\{ \Phi \right\} = [b] \left\{ \psi \right\} \tag{8.7}$$

it follows that

$$\left\{ \psi \right\} = [b]^{-1} \left\{ \Phi \right\} \tag{8.8}$$

and that

$$\left\{ \psi \right\}^T = \left\{ \Phi \right\}^T [b]^{-T} \tag{8.9}$$

and that

$$\left\{ \psi \right\}^T \left\{ A \right\} = \left\{ \Phi \right\}^T [b]^{-T} \left\{ A \right\} \tag{8.10}$$

This last result suggests the usefulness of defining the auxiliary vector

$$\left\{ A^* \right\} = [b]^{-T} \left\{ A \right\} \tag{8.11}$$

We now find that the basic relation to be solved, Eq. (8.1), can be rewritten in the alternative form

$$\left\{ \Phi \right\}^T \left\{ A^* \right\} = f(\theta) \tag{8.12}$$

The great advantage of (8.12) over the original form (8.1) lies in the

orthogonality properties of the functions Φ_n. For if we now multiply (8.12) through by any one of the functions $\Phi_n(\theta)$, then integrate over the semi-span, we find that, because of the orthogonality properties, all integrals but the one in n vanish on the left with the simple result that

$$A^*_n = \int_0^{\pi/2} f(\theta)\, \Phi_n(\theta)\, d\theta \tag{8.13}$$

$$n = 3, 5, 7, \text{---} (n-2)$$

Upon substituting into (8.13) the relation

$$\Phi_n = \left\{ b_n \right\}_n^T \left\{ \psi_n \right\} \tag{8.14}$$

and utilizing the definitions (8.5), this result can be rewritten in a form more convenient for calculation, namely,

$$A^*_n = \left\{ b_n \right\}_n^T \left\{ F \right\}_n \tag{8.15}$$

This can readily be generalized over the entire vector $\left\{ A^* \right\}$. The process amounts to pre-multiplying Eq. (8.12) through by (8.7), then integrating over the semi-span and applying the ortho-normal conditions in the usual way. We thereby obtain

$$\left\{ A^* \right\} = [b] \left\{ F \right\} \tag{8.16}$$

A significant property of the constants A^*_n as revealed by Eqs. (8.15) or (8.16) is that they are true invariants of the solution in the following sense. The value of A^*_n depends on F_3, F_5, F_7, --- F_n. But owing to the zeros in [b], it is independent of all the higher order terms F_{n+2}, F_{n+4}, and so on. Consequently, adding on additional terms to the solution in no wise affects the values of the constants A^*_n previously calculated.

Nevertheless, while $\left\{ A^* \right\}$ contains the true invariants of the solution, it is more convenient in some ways to continue to express the solution in terms of the original unknown constants, as contained in the vector $\left\{ A \right\}$. To solve for $\left\{ A \right\}$ we need merely to premultiply Eq. (8.11) through by $[b]^T$. The result is simply

$$\left\{ A \right\} = [b]^T \left\{ A^* \right\} \tag{8.17}$$

In many cases it may be preferred to bypass the vector $\left\{ A^* \right\}$ altogether in the solution. To show how this can be done, we simply substitute for $\left\{ A^* \right\}$ from (8.16) into (8.17). The result becomes, finally,

$$\left\{ A \right\} = [b]^T [b] \left\{ F \right\} \tag{8.18}$$

Thus, once the orthogonality matrix [b] is known, Eqs. (8.5) and (8.18) summarize the solution for the initially unknown constants A_n. When these are fixed, all remaining features of the solution can be established from the equations developed in the earlier sections.

The solution represented by Eq. (8.18), although it entails no explicit matrix inversion at any point, is in fact equivalent to a matrix inversion.

To show this, we premultiply the basic relation, Eq. (8.1), through by $\{\Psi\}$ then integrate over the semi-span and apply the definitions of J_{nk} and F_n as given in Eqs. (7.5) and (8.5) respectively. In this way we obtain

$$[J]\left\{A\right\} = \left\{F\right\} \tag{8.19}$$

The solution of this equation is

$$\left\{A\right\} = [J]^{-1}\left\{F\right\} \tag{8.20}$$

Comparison of Eq. (8.18) with (8.20) discloses that

$$[b]^T[b] = [J]^{-1} \tag{8.21}$$

which demonstrates the sense in which calculation of matrix [b] amounts to the inversion of matrix [J].

Bearing in mind that the elements of [b] are calculated in the first place solely from the elements of [J], it is clear that what we have here in effect is a particular method of inverting matrix [J]. But since [J] could be any symmetrical matrix, the method amounts to an inversion procedure applicable to any symmetrical matrix. While the method happened to be developed in this instance in terms of the orthogonal functions Φ_n, these functions do not actually occur in the resulting calculation formulas. Hence the final algorithm may be applied to invert any symmetrical matrix quite irrespective of its origins. In this particular instance, the elements of [J] happen to be certain specific integrals; the same inversion procedure

should also apply to symmetrical matrices whose elements are defined in some other way. Hence the technique is clearly applicable not just to the wing problem, but to a wide range of other applications as well.

One advantage of solving by means of the matrix [b] is that the calculations are strictly progressive and cumulative. The process of adding on each additional row to [b] provides additional finer detail about the solution without disrupting the details calculated from earlier rows. This becomes somewhat clearer if the solution be expressed in terms of the constants A_n^* rather than in terms of the original constants A_n. The invariance properties of the A_n^* have been noted earlier. In contrast with this, the constants A_n fluctuate each time a higher order term is added to the solution, and approach their true ultimate values only in the limit as the number of terms increases without bound. This analysis therefore sheds much light on the nature of the convergence process that, in principle, defines the exact mathematical solution of the original basic equation.

The solution may therefore be viewed as the result of superimposing an indefinite number of modes each of which is orthogonal to all of the preceding modes. To illustrate this concept, consider the solution for the additional lift distribution, Eq. (5.26), rewritten in matrix notation, that is,

$$\frac{c_{\ell_A}}{C_L}\left(\frac{c}{\bar{c}}\right) = \frac{4}{\pi}\sin\theta + \frac{4}{\pi}\left\{A\right\}^T\left\{\sin n\theta\right\} \tag{8.22a}$$

$$= \frac{4}{\pi} \sin\theta + \frac{4}{\pi} \left\{ A^* \right\}^T [b] \left\{ \sin n\theta \right\} \qquad (8.22b)$$

$$= \frac{4}{\pi} \sin\theta + \left\{ A^* \right\}^T \left\{ \lambda_n(\theta) \right\} \qquad (8.22c)$$

where

$$\left\{ \lambda_n(\theta) \right\} = \left\{ \begin{array}{c} \lambda_3(\theta) \\ \lambda_5(\theta) \\ \lambda_7(\theta) \\ --- \\ \lambda_N(\theta) \end{array} \right\} = \frac{4}{\pi} [b] \left\{ \begin{array}{c} \sin 3\theta \\ \sin 5\theta \\ \sin 7\theta \\ --- \\ \sin N\theta \end{array} \right\} \qquad (8.23)$$

Thus the functions $\lambda_3(\theta)$, $\lambda_5(\theta)$, $\lambda_7(\theta)$, --- $\lambda_N(\theta)$ as defined in Eq. (8.23) represent the successive invariant orthogonal modes, a linear combination of which fixes the additional lift distribution as shown in Eq. (8.22c). The addition of any mode n to the solution does not affect the amplitudes of the modes 3, 5, 7, -- (n-2) previously obtained. This example illustrates the theoretical concept, at any rate. Of course, for practical calculation, the ordinary form shown in Eq. (8.22a) would probably be preferred as involving somewhat less overall computation.

9. REFERENCES

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In this paper the standard wing equation, as normally derived from lifting line theory, is further refined and a solution procedure more basic than the usual collocation technique is developed. The calculation method adopted avoids the necessity of performing an explicit matrix inversion; all equations can be solved sequentially, one at a time. On the other hand this technique involves the evaluation of numerous integrals over the span. The calculations are cumulative, and can be carried as far as necessary to

achieve any required degree of accuracy. The analysis is interesting not only for purposes of practical calculation but also for the light it sheds on the essential mathematical structure of the basic aerodynamic phenomena involved. This same general method of calculation can also be readily adapted to the solution of other common types of engineering problems.